

# THE ORTHOGONAL CHARACTER TABLE OF $\mathrm{SL}_2(q)$

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ABSTRACT. The rational invariants of the  $\mathrm{SL}_2(q)$ -invariant quadratic forms on the real irreducible representations are determined. There is still one open question (see Remark 6.5) if  $q$  is an even square.

## 1. INTRODUCTION

Throughout the paper let  $G$  be a finite group. The isomorphism classes of  $\mathbb{C}G$ -modules are parametrized by their characters. Our aim is to extend this connection in order to also determine the  $G$ -invariant quadratic forms from the character table of  $G$ . The ordinary character table displays the characters  $\chi_V$  of the absolutely irreducible  $\mathbb{C}G$ -modules  $V$ . For each  $\chi_V$  let  $K$  be the maximal real subfield of the character field of  $V$  and  $W$  the irreducible  $KG$ -module such that  $V$  occurs in  $W \otimes_K \mathbb{C}$ . Then the space

$$\mathcal{F}_G(W) := \left\{ F : W \times W \rightarrow K \mid \begin{array}{l} F(v,w)=F(w,v) \text{ and} \\ F(gw,gv)=F(w,v) \text{ for all } g \in G, v,w \in W \end{array} \right\}$$

of  $G$ -invariant symmetric bilinear forms on  $W$  is at least one-dimensional and every non-zero  $F \in \mathcal{F}_G(W)$  is non-degenerate. The character  $\chi_V$  also determines the  $K$ -isometry classes of the elements of  $\mathcal{F}_G(W)$ . The orthogonal character table additionally contains the invariants (see Section 2) that determine the  $K$ -isometry classes of  $(W, F)$  for all non-zero  $F \in \mathcal{F}_G(W)$ .

For  $G = \mathrm{SL}_2(q)$  the ordinary character table was already known to Schur, [16]. This paper determines the orthogonal character tables of  $\mathrm{SL}_2(q)$  for all prime powers  $q$ . For  $q = 2^n$  with  $n$  even and the characters of degree  $q + 1$  we could not specify which even primes ramify in the Clifford algebra (see Section 6).

This work grew out of the first author's PhD thesis [2] written under the supervision of the second author. In this thesis, the first author also determines the ordinary orthogonal character tables for all (non-abelian) finite quasisimple groups of order up to 200,000.

## 2. INVARIANTS OF QUADRATIC SPACES

Let  $K$  be a field of characteristic 0,  $V$  an  $n$ -dimensional vector space over  $K$  and  $F : V \times V \rightarrow K$  a non-degenerate symmetric bilinear form. The two most important invariants attached to such a space  $(V, F)$  are the discriminant and the

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Clifford invariant.

The *discriminant* of  $(V, F)$  is

$$d_{\pm}(V, F) := (-1)^{n(n-1)/2} \det(V, F)$$

where the determinant  $\det(V, F) \in K/(K^{\times})^2$  is defined as the square class of the determinant of a Gram matrix of  $F$  with respect to any basis.

The Clifford algebra  $\mathcal{C}(V, F)$  is the quotient of the tensor algebra by the two-sided ideal  $\langle v \otimes v - \frac{1}{2}F(v, v) \cdot 1 \mid v \in V \rangle$ . A  $K$ -basis of  $\mathcal{C}(V, F)$  is given by the ordered tensors  $(\overline{b_{i_1} \otimes \dots \otimes b_{i_k}} \mid 1 \leq i_1 < \dots < i_k \leq n)$  of any basis  $(b_1, \dots, b_n)$  of  $V$ , in particular  $\dim(\mathcal{C}(V, F)) = 2^n$ . Put

$$c(V, F) := \begin{cases} \mathcal{C}(V, F) & \text{if } n \text{ is even,} \\ \mathcal{C}_0(V, F) := \langle \overline{b_{i_1} \otimes \dots \otimes b_{i_k}} \mid k \text{ even} \rangle & \text{if } n \text{ is odd.} \end{cases}$$

Then  $c(V, F) \cong \mathcal{D}^{r \times r}$  is a central simple  $K$ -algebra with involution and therefore it has order 1 or 2 in the Brauer group. The *Clifford invariant* of  $(V, F)$  is defined as the Brauer class of  $c(V, F)$ :

$$\mathfrak{c}(V, F) := [c(V, F)] = [\mathcal{D}] \in \text{Br}(K).$$

A more detailed exposition of this material may be found e.g. in [15].

Our interest in these two isometry invariants of quadratic spaces is mainly due to the following classical result by Helmut Hasse.

**Theorem 2.1** ([7]). *Over a number field  $K$  the isometry class of a quadratic space is uniquely determined by its dimension, its determinant, its Clifford invariant and its signature at all real places of  $K$ .*

We are mainly interested in the case where  $K$  is a number field. Then  $\mathcal{D}$  is either  $K$  or a quaternion division algebra over  $K$ . We use two notations for these  $\mathcal{D}$ , either as a symbol algebra or by giving all the local invariants of  $\mathcal{D}$ :

**Definition 2.2.** For  $a, b \in K$  let  $(a, b) := [(\frac{a, b}{K})] \in \text{Br}(K)$  where

$$\left(\frac{a, b}{K}\right) := \langle 1, i, j, k \mid i^2 = a, j^2 = b, ij = -ji = k \rangle.$$

By the Theorem of Hasse, Brauer, Noether, Albert (see [14, Theorem (32.11)]) any quaternion algebra  $\mathcal{D}$  over  $K$  is determined by the set of places  $\wp_1, \dots, \wp_s$  (the ramified places) of  $K$ , for which the completion of  $\mathcal{D}$  stays a division algebra. Therefore we also describe  $\mathcal{D} = \mathcal{Q}_{\wp_1, \dots, \wp_s}$  by its ramified places, where we assume that the center  $K$  is clear from the context.

**Example 2.3.** Let  $(V, F)$  be a bilinear space and  $a \in K^{\times}$ . Then the scaled space  $(V, aF)$  has the following algebraic invariants (see [10, 5.(3.16)] for the Clifford invariant):

$$d_{\pm}(V, aF) = \begin{cases} d_{\pm}(V, F) & \text{if } \dim(V) \text{ is even,} \\ ad_{\pm}(V, F) & \text{if } \dim(V) \text{ is odd.} \end{cases}$$

and

$$\mathfrak{c}(V, aF) = \begin{cases} \mathfrak{c}(V, F)(a, d_{\pm}(V, F)) & \text{if } \dim(V) \text{ is even,} \\ \mathfrak{c}(V, F) & \text{if } \dim(V) \text{ is odd.} \end{cases}$$

If

$$(V, F) = (V_1, F_1) \perp (V_2, F_2)$$

is the orthogonal direct sum of two subspaces the determinant is just the product  $\det(V, F) = \det(V_1, F_1) \cdot \det(V_2, F_2)$ .

The behavior of the Clifford invariant is more complicated, cf. [10]:  $\mathfrak{c}(V, F) =$

$$\begin{cases} \mathfrak{c}(V_1, F_1)\mathfrak{c}(V_2, F_2)(d_{\pm}(V_1, F_1), d_{\pm}(V_2, F_2)), & \dim(V_1) \equiv \dim(V_2) \pmod{2}, \\ \mathfrak{c}(V_1, F_1)\mathfrak{c}(V_2, F_2)(-d_{\pm}(V_1, F_1), d_{\pm}(V_2, F_2)), & \dim(V) \equiv \dim(V_1) \equiv 1 \pmod{2}. \end{cases}$$

**Example 2.4.** Let  $\mathbb{I}_n$  be the  $n$ -dimensional  $\mathbb{Q}$ -vector space that has an orthonormal basis  $(e_1, \dots, e_n)$ . Then  $d_{\pm}(\mathbb{I}_n) = (-1)^{n(n-1)/2}(\mathbb{Q}^{\times})^2$  and

$$\mathfrak{c}(\mathbb{I}_n) = \begin{cases} (1, 1) & n \equiv 0, 1, 2, 7 \pmod{8}, \\ (-1, -1) & n \equiv 3, 4, 5, 6 \pmod{8}. \end{cases}$$

The space  $\mathbb{A}_{n-1} := \langle \sum_{i=1}^n e_i \rangle^{\perp} \leq \mathbb{I}_n$  is the orthogonal complement of a space of discriminant  $n$  in  $\mathbb{I}_n$ . This allows to compute the discriminant and Clifford invariant of  $\mathbb{A}_{n-1}$  using the formulas from the previous example:  $d_{\pm}(\mathbb{A}_{n-1}) = (-1)^{(n-1)(n-2)/2}n(\mathbb{Q}^{\times})^2$  and  $\mathfrak{c}(\mathbb{A}_{n-1})$  depends on the value of  $n$  modulo 8:

$n \pmod{8}$	0, 1	2, 3	4, 5	6, 7
$\mathfrak{c}(\mathbb{A}_{n-1})$	1	$(-1, n)$	$(-1, -1)$	$(-1, -n)$

### 3. METHODS

**3.1. Orthogonal character tables.** Let  $\chi$  be a complex irreducible character of the finite group  $G$  and let  $K = \mathbb{Q}(\chi)^+$  be the maximal real subfield of the character field  $\mathbb{Q}(\chi)$ . Let  $V$  be the irreducible  $\mathbb{C}G$ -module affording the character  $\chi$  and let  $W$  be the irreducible  $KG$ -module such that  $V$  is a constituent of  $W_{\mathbb{C}} := \mathbb{C} \otimes_K W$ . Put

$$\mathcal{F}_G(W) := \left\{ F : W \times W \rightarrow K \mid \begin{array}{l} F(v, w) = F(w, v) \text{ and} \\ F(gw, gv) = F(w, v) \text{ for all } g \in G, v, w \in W \end{array} \right\}$$

the space of  $G$ -invariant symmetric bilinear forms on  $W$ . As  $W$  is irreducible, all non-zero elements of  $\mathcal{F}_G(W)$  are non-degenerate and an easy averaging argument shows that  $\mathcal{F}_G(W)$  always contains a totally positive definite form  $F_0$ . We call  $W$  *uniform* if  $\mathcal{F}_G(W) = \langle F_0 \rangle_K$  is one-dimensional over  $K$ .

**Remark 3.1.** There are three different situations to be considered:

- (a)  $K = \mathbb{Q}(\chi)$  and  $V = W_{\mathbb{C}}$ : Then  $W$  is an absolutely irreducible  $KG$ -module and hence uniform.
- (b)  $K = \mathbb{Q}(\chi)$  and  $W_{\mathbb{C}} \cong V \oplus V$ : Then the Schur index of  $\chi$  over  $K$  is 2,  $\chi(1)$  is even, and [18] tells us that  $d_{\pm}(F) \in (K^{\times})^2$  for all non-zero  $F \in \mathcal{F}_G(W)$ . If the real Schur index of  $\chi$  is one, then  $\dim(\mathcal{F}_G(W)) = 3$ .

If the real Schur index of  $\chi$  is 2, then  $W$  is uniform and [18, Theorem B] also gives the Clifford invariant of  $(W, F)$ :

$$\mathfrak{c}(W, F) = \begin{cases} 1 & \text{if } \dim_K(W) \equiv 0 \pmod{8} \\ [\text{End}_{KG}(W)] & \text{if } \dim_K(W) \equiv 4 \pmod{8}. \end{cases}$$

- (c)  $[\mathbb{Q}(\chi) : K] = 2$ . Then  $\chi_W = m(\chi + \overline{\chi})$  for some  $m \in \mathbb{N}$  and  $W$  is uniform if and only if  $m = 1$ . Choose  $\delta \in K$  such that  $\mathbb{Q}(\chi) = K(\sqrt{\delta})$ , then  $d_{\pm}(F) = \delta^{m\chi(1)}(K^{\times})^2$  for all  $0 \neq F \in \mathcal{F}_G(W)$  (see [15, Theorem 10.1.4], [2, Theorem 4.3.9]).

**Definition 3.2.** Let  $\chi$ ,  $K := \mathbb{Q}(\chi)^+$ ,  $W$  be as above. Put  $n := \dim_K(W)$  and choose  $0 \neq F \in \mathcal{F}_G(W)$ . If  $n$  is even then we define

$$d_{\pm}(\chi) := d_{\pm}(W, F).$$

If  $n$  is odd, or  $n$  is even,  $W$  is uniform, and  $d_{\pm}(\chi) = 1$ , then we put

$$\mathfrak{c}(\chi) := \mathfrak{c}(W, F).$$

The orthogonal character table of  $G$  is the complex character table of  $G$  with this additional information added.

By Example 2.3 and Remark 3.1 the values  $d_{\pm}(\chi)$  and  $\mathfrak{c}(\chi)$  are well defined, i.e. independent of the choice of the non-zero  $F \in \mathcal{F}_G(W)$ .

**3.2. Clifford orders.** Let us now assume that  $K$  is a local or global field of characteristic 0, i.e.  $K$  is a finite extension of either  $\mathbb{Q}_p$  or  $\mathbb{Q}$ , and let  $R$  denote the ring of integers in  $K$ . Let  $V$  be a finite dimensional vector space over  $K$  and  $F : V \times V \rightarrow K$  a symmetric bilinear form with associated quadratic form

$$Q_F : V \rightarrow K, v \mapsto Q_F(v) = \frac{1}{2}F(v, v).$$

**Definition 3.3.** A lattice  $L$  in  $V$  is a finitely generated  $R$ -submodule of  $V$  that contains a  $K$ -basis of  $V$ . The lattice  $L$  is called integral, if  $F(L, L) \subseteq R$  and even, if  $Q_F(L) \subseteq R$ . The dual lattice of  $L$  is  $L^{\#} := \{v \in V \mid F(v, L) \subseteq R\}$  and  $L$  is called unimodular, if  $L = L^{\#}$ .

Even unimodular lattices are called regular quadratic  $R$ -modules in [9]. If  $2 \notin R^{\times}$ , then there are no regular  $R$ -modules  $L$  of odd dimension. Kneser calls an even lattice  $L$  of odd dimension such that  $L^{\#}/L \cong R/2R$  a semi-regular quadratic  $R$ -module.

**Theorem 3.4** ([9, Satz 15.8]). Assume that  $R$  is a complete discrete valuation ring (with finite residue class field) and let  $L$  be a regular or semi-regular quadratic  $R$ -module in  $(V, Q_F)$ . If  $\dim(V) \geq 3$  then  $L \cong \mathbb{H}(R) \perp M$  for some regular or semi-regular quadratic  $R$ -module  $M$ . Here  $\mathbb{H}(R)$  is the hyperbolic plane, the regular free  $R$ -lattice with basis  $(e, f)$  such that  $Q_F(e) = Q_F(f) = 0$  and  $F(e, f) = 1$ .

As both invariants, the Clifford invariant and the discriminant of the hyperbolic plane  $\mathbb{H}(K) = K\mathbb{H}(R)$  are trivial, we obtain the following corollary.

**Corollary 3.5.** *Under the assumption of the theorem let  $\dim(V)$  be odd and  $L$  be a semi-regular lattice in  $V$ . Then  $\mathfrak{c}(V, F) = 1$ .*

*Proof.* We proceed by induction on the dimension of  $V$ . If  $\dim(V) = 1$  then  $c(V, F) = K$  and so  $\mathfrak{c}(V, F) = 1$ . So assume that  $\dim(V) \geq 3$ . Then  $L \cong \mathbb{H}(R) \perp M$  and hence  $V \cong \mathbb{H}(K) \perp KM$  for some semi-regular lattice  $M$  in  $KM$ . By induction we have  $\mathfrak{c}(KM, F|_{KM}) = 1$ . So

$$\mathfrak{c}(V, F) = \mathfrak{c}(KM, F|_{KM})\mathfrak{c}(\mathbb{H}(K))(-d_{\pm}(KM, F|_{KM}), d_{\pm}(\mathbb{H}(K))) = 1. \quad \square$$

**Remark 3.6.** *Let  $L$  be an even lattice in  $V$ . Then the Clifford order  $\mathcal{C}(L, F)$  of  $L$  is defined to be the  $R$ -subalgebra of  $\mathcal{C}(V, F)$  generated by  $L$ . As  $Q_F(L) \subseteq R$ , the Clifford order is an  $R$ -lattice in  $\mathcal{C}(V, F)$ , in particular finitely generated over  $R$ . If  $L$  has an orthogonal basis  $(b_1, \dots, b_n)$ , then the ordered tensors  $(\overline{b_{i_1} \otimes \dots \otimes b_{i_k}} \mid 1 \leq i_1 < \dots < i_k \leq n)$  form an  $R$ -basis of  $\mathcal{C}(L, F)$ . In this case it is easy to compute the determinant of  $\mathcal{C}(L, F)$  and of  $\mathcal{C}_0(L, F)$  with respect to the reduced trace bilinear form (see [2, Theorem 7.2.2]): Up to some power of 2 they are both powers of  $Q_F(b_1) \cdots Q_F(b_n)$ .*

**Corollary 3.7.** *Assume that  $K$  is a number field,  $2 \neq p \in \mathbb{Z}$  is some odd prime and  $\wp$  is a prime ideal of  $K$  containing  $p$ . Denote the completion of  $K$  at  $\wp$  by  $K_{\wp}$  and its valuation ring by  $R_{\wp}$ . Assume that there is a lattice  $L$  in  $V$  such that  $L_{\wp} = R_{\wp} \otimes L$  is an even unimodular  $R_{\wp}$ -lattice. Then*

$$[c(V, F) \otimes K_{\wp}] = 1 \in \text{Br}(K_{\wp}).$$

*Proof.* Since  $2 \in R_{\wp}^{\times}$  the lattice  $L_{\wp}$  has an orthogonal basis and Remark 3.6 shows that the determinant of the Clifford order  $\mathcal{C}(L_{\wp}, F)$  and also of  $\mathcal{C}_0(L_{\wp}, F)$  is a unit in  $R_{\wp}$ . In particular the determinant of a maximal order in  $c(V, F) \otimes K_{\wp}$  is a unit in  $R_{\wp}$ , which shows that this central simple  $K_{\wp}$ -algebra is a matrix ring over  $K_{\wp}$  (see for instance [14, Theorem (20.3)]).  $\square$

A bit more generally we may also compute the Clifford invariant of a bilinear space that contains a lattice of prime determinant:

**Corollary 3.8.** *Keep the assumptions of Corollary 3.7 and let  $(W_{\wp}, E_{\wp})$  be a 1-dimensional bilinear  $K_{\wp}$  vector space such that the  $\wp$ -adic valuation of the discriminant of  $E_{\wp}$  is odd. Then*

$$\mathfrak{c}((V \otimes K_{\wp}, F) \perp (W_{\wp}, E_{\wp})) = 1 \in \text{Br}(K_{\wp}) \text{ if and only if } d_{\pm}(V \otimes K_{\wp}, F) \in (K_{\wp}^{\times})^2.$$

*Proof.* Clearly the Clifford invariant of the 1-dimensional space is trivial and also  $\mathfrak{c}(V \otimes K_{\wp}, F)$  is trivial by Corollary 3.7. So the formula in Example 2.3 gives us the Clifford invariant of the orthogonal sum as

$$\mathfrak{c}((V \otimes K_{\wp}, F) \perp (W_{\wp}, E_{\wp})) = (d_{\pm}(V \otimes K_{\wp}, F), u\pi)$$

where  $u$  is a unit and  $\pi$  is a prime element in the valuation ring  $R_{\wp}$ . As  $d := d_{\pm}(V \otimes K_{\wp}, F) \in R_{\wp}^{\times}$ , this quaternion symbol is trivial if and only if  $d$  is a square.  $\square$

**3.3. A Clifford theory of orthogonal representations.** Let  $N \trianglelefteq G$  be a normal subgroup. Clifford theory explains the interplay between irreducible representations of  $N$  and  $G$  (see for instance [4, Section 11.1]). We want to describe the behavior of invariant forms under this correspondence.

Let  $K$  be a totally real number field and  $V$  an irreducible  $KG$ -module with a non-degenerate invariant form  $F$ . We will then call  $(V, F)$  an *orthogonal representation* of  $G$ . Let  $U$  be an irreducible  $KN$ -module occurring as a direct summand of  $V|_N$  with multiplicity  $e$ . Let  $I$  be the inertia group of  $U$ , of index  $t := [G : I]$  in  $G$ , and let  $G = \bigsqcup_{i=1}^t g_i I$  be a decomposition of  $G$  into left  $I$ -cosets. We then have the following decomposition of  $V|_N$  into pairwise non-isomorphic irreducible  $KN$ -modules  ${}^{g_i}U$  ( $i = 1, \dots, t$ ):

$$(1) \quad V|_N \cong \bigoplus_{i=1}^t ({}^{g_i}U)^e,$$

In this situation we obtain the following theorem

**Lemma 3.9.** *The decomposition (1) is orthogonal*

$$(V|_N, F) = ({}^{g_1}U^e, F_1) \perp ({}^{g_2}U^e, F_2) \perp \dots \perp ({}^{g_t}U^e, F_t)$$

and the forms  $F_i$  are non-degenerate and pairwise  $K$ -isometric.

*Proof.* Clearly, the restriction of  $F$  to the direct summand  ${}^{g_i}U^e$  is  $N$ -invariant. For  $i \neq j$  we have

$${}^{g_i}U \cong {}^{g_i}U^* \not\cong {}^{g_j}U$$

so the summands  $({}^{g_i}U)^e$  are orthogonal to each other and the  $F_i$  are non-degenerate. The elements  $g_j^{-1}g_i \in G \leq O(V, F)$  induce isometries between  $F_i$  and  $F_j$ .  $\square$

**Example 3.10.** Consider an odd prime  $p$ , a natural number  $n$  and abbreviate  $q := p^n$ . Let  $C_{(q-1)/2} \cong H \leq \mathrm{GL}_n(\mathbb{F}_p)$  be a subgroup acting with regular orbits on  $\mathbb{F}_p^n \setminus \{0\}$  in its natural action. Then the group  $G := C_p^n \rtimes H$ , which is isomorphic to the normalizer of a Sylow  $p$ -subgroup in  $\mathrm{PSL}_2(q)$  has  $(q-1)/2$  linear characters and two non-linear characters  $\psi_1, \psi_2$  of degree  $(q-1)/2$  with Schur index 1 and character field

$$\mathbb{Q}(\psi_1) = \mathbb{Q}(\psi_2) = \begin{cases} \mathbb{Q}(\sqrt{q}) & \text{if } q \equiv 1 \pmod{4}, \\ \mathbb{Q}(\sqrt{-q}) & \text{if } q \equiv 3 \pmod{4}. \end{cases}$$

Let  $H_1$  be the unique subgroup of  $H$  of order  $\frac{p-1}{2}$  and put  $N := C_p^n \rtimes H_1$ . Then  $N \trianglelefteq G$  and we will apply Theorem 3.9 to this normal subgroup in order to compute the discriminant  $d_{\pm}(\psi_i)$  in the case  $q \equiv 1 \pmod{4}$ .

Let  $\psi \in \{\psi_1, \psi_2\}$ ,  $K = \mathbb{Q}(\psi) = \mathbb{Q}(\sqrt{q})$  and  $(V, F)$  an orthogonal  $KG$ -module whose character is  $\psi$ .

There is a character  $\mathbf{1} \neq \chi \in \mathrm{Irr}(C_p^n)$  such that  $\psi = \mathrm{ind}_{C_p^n}^G(\chi) = \mathrm{ind}_N^G(\mathrm{ind}_{C_p^n}^N(\chi))$ . Ordinary Clifford theory shows that  $\kappa := \mathrm{ind}_{C_p^n}^N(\chi)$  is irreducible and an easy

application of Frobenius reciprocity reveals  $(\psi|_N, \kappa)_N = 1$ .

Thus we obtain an orthogonal decomposition

$$(V|_N, F) \cong (V_1, F_1) \perp \dots \perp (V_t, F_t)$$

where  $F_1 \cong \dots \cong F_t$  by Lemma 3.9. We have  $t = \frac{1}{2} \frac{q-1}{p-1}$  if  $K = \mathbb{Q}$  and  $t = \frac{q-1}{p-1}$  if  $K = \mathbb{Q}(\sqrt{p})$ .

Notice that  $\kappa$  is a faithful character of a group isomorphic to  $C_p \rtimes C_{\frac{p-1}{2}}$ . As the trace forms of cyclotomic fields are well understood (cf. [11, Section 3.3.2]), we can find the determinants of the  $(V_i, F_i)$  as

$$\det(V_i, F_i) = \det(V_1, F_1) = \begin{cases} p(\mathbb{Q}^\times)^2 & \text{if } n \text{ is even} \\ u\sqrt{p}(\mathbb{Q}(\sqrt{p})^\times)^2 & \text{if } n \text{ is odd} \end{cases}$$

for some unit  $u$  of the ring of integers of  $\mathbb{Q}(\sqrt{p})$ . In conclusion, we obtain

$$\det(\psi) = \begin{cases} 1(\mathbb{Q}^\times)^2 & \text{if } n \equiv 0 \pmod{4} \text{ or } p \equiv 3 \pmod{4}, \\ p(\mathbb{Q}^\times)^2 & \text{if } n \equiv 2 \pmod{4} \text{ and } p \equiv 1 \pmod{4}, \\ u\sqrt{p}(\mathbb{Q}(\sqrt{p})^\times)^2 & \text{if } n \equiv 1 \pmod{2}, \end{cases}$$

In the case  $q \equiv 3 \pmod{4}$  the character  $\psi$  has non-real values and we find  $d_\pm(\psi) = -p(\mathbb{Q}^\times)^2$ .

#### 4. THE ORTHOGONAL CHARACTER TABLE OF $\mathrm{SL}_2(q)$ FOR ODD $q$

Let  $p$  be an odd prime,  $n$  a natural number, put  $q := p^n$  and let  $G := \mathrm{SL}_2(q)$  be the group of all  $2 \times 2$  matrices of determinant 1 over the field with  $q$  elements. A reference for the ordinary (and modular) representation theory of this group is, for example [1]. We use the ordinary character table and the notation of the absolutely irreducible characters from [5]:

**Theorem 4.1** ([5, Theorem 38.1]). *Let  $\langle \nu \rangle = \mathbb{F}_q^\times$ . Consider*

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad z = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} 1 & 0 \\ \nu & 1 \end{pmatrix}, \quad a = \begin{pmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{pmatrix}$$

and let  $b \in \mathrm{SL}_2(q)$  be an element of order  $q+1$ .

For  $x \in \mathrm{SL}_2(q)$ , let  $(x)$  denote the conjugacy class containing  $x$ .  $\mathrm{SL}_2(q)$  has the following  $q+4$  conjugacy classes of elements, listed together with the size of the classes.

$x$	$1$	$z$	$c$	$d$	$zc$	$zd$	$a^\ell$	$b^m$
$ (x) $	$1$	$1$	$\frac{1}{2}(q^2-1)$	$\frac{1}{2}(q^2-1)$	$\frac{1}{2}(q^2-1)$	$\frac{1}{2}(q^2-1)$	$q(q+1)$	$q(q-1)$

where  $1 \leq \ell \leq \frac{q-3}{2}$ ,  $1 \leq m \leq \frac{q-1}{2}$ .

Put

$$\varepsilon := (-1)^{(q-1)/2}, \quad \zeta_r := \exp(2\pi i/r) \text{ and } \vartheta_r^{(s)} := \zeta_r^s + \zeta_r^{-s} \text{ for } r, s \in \mathbb{N}.$$

Then the character table of  $\mathrm{SL}_2(q)$  reads as

	1	$z$	$c$	$d$	$a^\ell$	$b^m$
$\mathbb{1}$	1	1	1	1	1	1
$\psi$	$q$	$q$	0	0	1	-1
$\chi_i$	$q+1$	$(-1)^i(q+1)$	1	1	$\vartheta_{q-1}^{(i\ell)}$	0
$\theta_j$	$q-1$	$(-1)^j(q-1)$	-1	-1	0	$-\vartheta_{q+1}^{(jm)}$
$\xi_1$	$\frac{1}{2}(q+1)$	$\frac{1}{2}\varepsilon(q+1)$	$\frac{1}{2}(1+\sqrt{\varepsilon q})$	$\frac{1}{2}(1-\sqrt{\varepsilon q})$	$(-1)^\ell$	0
$\xi_2$	$\frac{1}{2}(q+1)$	$\frac{1}{2}\varepsilon(q+1)$	$\frac{1}{2}(1-\sqrt{\varepsilon q})$	$\frac{1}{2}(1+\sqrt{\varepsilon q})$	$(-1)^\ell$	0
$\eta_1$	$\frac{1}{2}(q-1)$	$-\frac{1}{2}\varepsilon(q-1)$	$\frac{1}{2}(-1+\sqrt{\varepsilon q})$	$\frac{1}{2}(-1-\sqrt{\varepsilon q})$	0	$(-1)^{m+1}$
$\eta_2$	$\frac{1}{2}(q-1)$	$-\frac{1}{2}\varepsilon(q-1)$	$\frac{1}{2}(-1-\sqrt{\varepsilon q})$	$\frac{1}{2}(-1+\sqrt{\varepsilon q})$	0	$(-1)^{m+1}$

where  $1 \leq i \leq \frac{q-3}{2}$ ,  $1 \leq j \leq \frac{q-1}{2}$ ,  $1 \leq \ell \leq \frac{q-3}{2}$ ,  $1 \leq m \leq \frac{q-1}{2}$ .

The columns for the classes  $(zc)$  and  $(zd)$  are omitted because for any irreducible character  $\chi$  the relation  $\chi(zc) = \frac{\chi(z)}{\chi(1)}\chi(c)$  holds.

**Theorem 4.2.** The following table gives the orthogonal character table of  $\mathrm{SL}_2(q)$ .

$\chi$	$K$	$\dim_K(W)$	$\mathbf{c}(\chi)$	$\mathbf{d}_\pm(\chi)$	$q$
$\mathbb{1}$	$\mathbb{Q}$	1	1	—	<i>all</i>
$\psi$	$\mathbb{Q}$	$q$	$\mathbf{c}(\mathbb{A}_q)$	—	<i>all</i>
$\chi_i$ <i>i even</i>	$\mathbb{Q}(\vartheta_{q-1}^{(i)})$	$q+1$	—	$\varepsilon(\vartheta_{q-1}^{(2i)} - 2)q$	<i>all</i>
$\chi_i$ <i>i odd</i>	$\mathbb{Q}(\vartheta_{q-1}^{(i)})$	$2(q+1)$	$[\mathrm{End}_{KG}(W)]$ 1	1 1	1 (mod 4) 3 (mod 4)
$\theta_j$ <i>j even</i>	$\mathbb{Q}(\vartheta_{q+1}^{(j)})$	$q-1$	1 if $q = \square$	$\varepsilon q$	<i>all</i>
$\theta_j$ <i>j odd</i>	$\mathbb{Q}(\vartheta_{q+1}^{(j)})$	$2(q-1)$	1 $[\mathrm{End}_{KG}(W)]$	1 1	1 (mod 4) 3 (mod 4)
$\xi_1, \xi_2$	$\mathbb{Q}(\sqrt{q})$	$\frac{q+1}{2}$	1 $[\mathcal{Q}_{p,\infty} \otimes K]$	— —	$q \equiv 1, -3 \pmod{16}$ $q \equiv 5, 9 \pmod{16}$
$\xi_1 = \overline{\xi_2}$	$\mathbb{Q}$	$q+1$	1 $(-1, -1)$	1 1	3 (mod 8) 7 (mod 8)
$\eta_1, \eta_2$	$\mathbb{Q}(\sqrt{q})$	$q-1$	$[\mathcal{Q}_{p,\infty} \otimes K]$	1	1 (mod 4)
$\eta_1 = \overline{\eta_2}$	$\mathbb{Q}$	$q-1$	—	$-q$	3 (mod 4)

We use the abbreviations introduced in Theorem 4.1. As before  $K$  is the maximal real subfield of the character field and  $W$  the irreducible  $KG$ -module, whose character contains  $\chi$ .



## 5. THE PROOF OF THEOREM 4.2

**5.1. The faithful characters of  $G$ .** The faithful irreducible characters of  $\mathrm{SL}_2(q)$  either have real Schur index 2 or they take values in an imaginary quadratic number field. Janusz [8, Theorem] contains an explicit description of the endomorphism rings  $\mathrm{End}_{KG}(W)$ . In particular their discriminants and Clifford invariants can be read off from Remark 3.1 (b) and (c).

**5.2. The non-faithful characters  $\eta_i$ .** If  $q \equiv 3 \pmod{4}$  then the characters  $\eta_1$  and  $\eta_2$  of degree  $(q-1)/2$  have character field  $\mathbb{Q}(\sqrt{\epsilon q}) = \mathbb{Q}(\sqrt{-p})$  and Schur index 1. So Remark 3.1 (c) yields their discriminant.

**5.3. The Steinberg character.** The character  $\psi$  is a non-faithful character of degree  $q$ . As  $\mathbb{1} + \psi$  is the character of a 2-transitive permutation representation of  $G$ , the invariants of  $\psi$  are those of  $\mathbb{A}_q$  as given in Example 2.4.

**5.4. The characters  $\theta_j$ ,  $j$  even.** For even  $j$ , the character  $\theta_j$  is a non-faithful character of even degree  $q-1$  with totally real character field  $K$  and Schur index 1. Let  $(W, F)$  be the orthogonal  $KG$ -module affording the character  $\theta_j$ . Then the restriction of  $W$  to the Borel subgroup  $B \cong (C_p)^n \rtimes C_{(q-1)/2}$  of  $\mathrm{PSL}_2(q)$  has character  $\psi_1 + \psi_2$  from Example 3.10. As  $d_{\pm}(\psi_1)$  and  $d_{\pm}(\psi_2)$  are Galois conjugate, the formula for  $d_{\pm}(\psi_1)$  in Example 3.10 yields

$$d_{\pm}(\theta_j) = \begin{cases} 1(K^{\times})^2 & n \text{ even} \\ \epsilon p(K^{\times})^2 & n \text{ odd.} \end{cases}$$

If  $n$  is even then we can also deduce the Clifford invariant of  $(W, F)$ : In this case  $q \equiv 1 \pmod{4}$  so  $-\zeta_{q+1}^2$  is a primitive  $q+1$ st root of unity and hence all characters of degree  $q-1$  of the group  $\mathrm{PSL}_2(q)$  extend to characters of  $\mathrm{PGL}_2(q)$  with the same character field (see [17, Table III] for a character table) and of Schur index 1 (see [6]). So  $(W, F)$  is also an orthogonal representation of  $\mathrm{PGL}_2(q)$  and restricting  $(W, F)$  to  $B$ , we obtain the orthogonal sum of two isometric spaces  $(W, F) \cong (V_1, F_1) \perp (V_2, F_2)$  because the normalizer of  $B$  in  $\mathrm{PGL}_2(q)$  interchanges  $V_1$  and  $V_2$ . By Example 3.10 we have  $d_{\pm}(V_i, F_i) = p$  if  $n \equiv 2 \pmod{4}$  and  $p \equiv 1 \pmod{4}$  and  $d_{\pm}(V_i, F_i) = 1$  otherwise ( $i = 1, 2$ ). In both cases  $(d_{\pm}(V_1, F_1), d_{\pm}(V_2, F_2)) = 1 \in \mathrm{Br}(\mathbb{Q})$  and so by Example 2.3  $\mathfrak{c}(W, F) = \mathfrak{c}(V_1, F_1)\mathfrak{c}(V_2, F_2) = \mathfrak{c}(V_1, F_1)^2 = 1$ .

**5.5. The characters  $\chi_i$ ,  $i$  even.** For even  $i$ , the character  $\chi_i$  is a non-faithful character of even degree  $q+1$  with totally real character field  $K$  and Schur index 1. As before we restrict  $\chi_i$  to the Borel subgroup and obtain

$$\chi_i|_B = \psi_1 + \psi_2 + \alpha + \bar{\alpha}$$

where  $\psi_1, \psi_2$  are as in 5.4 and  $\alpha$  is a complex linear character of  $B$ . Comparing character values we obtain that  $\alpha(y) = \zeta_{q-1}^i$  for a suitably chosen generator  $y$  of  $C_{(q-1)/2} \leq B$ . In particular  $\mathbb{Q}(\alpha) = \mathbb{Q}(\zeta_{q-1}^i) = K(\sqrt{\vartheta_{q-1}^{(2i)} - 2})$  and hence Remark 3.1 (c) tells us that  $d_{\pm}(\alpha) = \vartheta_{q-1}^{(2i)} - 2$ . The discriminant of  $\psi_1$  and  $\psi_2$  behave as in 5.4 and hence we compute the discriminant  $d_{\pm}(\chi_i) = \epsilon(\vartheta_{q-1}^{(2i)} - 2)q$ .

**5.6. The characters  $\xi_1, \xi_2$  for  $q \equiv 1 \pmod{4}$ .** Assume that  $q = p^n \equiv 1 \pmod{4}$ . Then the two characters  $\xi_1$  and  $\xi_2$  of odd degree  $\frac{q+1}{2}$  factor through  $\mathrm{PSL}_2(q)$  and have a totally real character field  $K = \mathbb{Q}(\chi_1) = \mathbb{Q}(\chi_2) = \mathbb{Q}(\sqrt{q})$ .

**Proposition 5.1.** *There are the following two possibilities for  $\mathfrak{c}(\xi_1) = \mathfrak{c}(\xi_2)$ :*

	$n \text{ even}$		$n \text{ odd}$			
$q \pmod{16}$	1	9	1	-3	9	5
$\star \mathfrak{c}(\xi_1) = \mathfrak{c}(\xi_2)$	1	$[\mathcal{Q}_{p,\infty}]$	1	1	$[\mathcal{Q}_{\infty_1,\infty_2}]$	$[\mathcal{Q}_{\infty_1,\infty_2}]$
$\mathfrak{c}(\xi_1) = \mathfrak{c}(\xi_2)$	$[\mathcal{Q}_{2,p}]$	$[\mathcal{Q}_{2,\infty}]$	$[\mathcal{Q}_{\wp_1,\wp_2}]$	$[\mathcal{Q}_{2,\sqrt{p}}]$	$[\mathcal{Q}_{\infty_1,\infty_2,\wp_1,\wp_2}]$	$[\mathcal{Q}_{\infty_1,\infty_2,2,\sqrt{p}}]$

Here, for  $p \equiv 1 \pmod{8}$  and  $n$  odd,  $\wp_1$  and  $\wp_2$  denote the two places of  $K = \mathbb{Q}(\sqrt{p})$  that divide 2.

*Proof.* Let  $\xi$  be one of  $\xi_1$  or  $\xi_2$ ,  $K = \mathbb{Q}(\sqrt{q})$  and  $W$  the  $KG$ -module affording the character  $\xi$ . Since  $\mathcal{F}_G(W)$  always contains a totally positive definite form, we know that  $\mathfrak{c}(\xi) \otimes \mathbb{R} = 1$  if  $q \equiv 1, -3 \pmod{16}$  and  $\mathfrak{c}(\xi) \otimes \mathbb{R} \neq 1$  otherwise, for all real places of  $K$ . If  $K \neq \mathbb{Q}$  then  $\xi_1$  and  $\xi_2$  are Galois conjugate and so are  $\mathfrak{c}(\xi_1)$  and  $\mathfrak{c}(\xi_2)$ . The outer automorphism of  $G$  interchanges  $\xi_1$  and  $\xi_2$  which also shows that  $\mathfrak{c}(\xi_1) = \mathfrak{c}(\xi_2)$ , so this algebra is stable under the Galois group of  $K$ . Moreover the only possible finite primes of  $K$  that ramify in  $\mathfrak{c}(\xi)$  are those dividing  $p$  or 2. This is seen as follows: The representation  $\xi$  is irreducible modulo all other primes  $\ell$  (see [1, Section 9.3]) so in particular there is a  $G$ -invariant lattice  $L$  in  $W$  whose determinant is not divisible by  $\ell$  and hence  $\ell$  does not ramify in  $\mathfrak{c}(W, F)$  by Remark 3.6. Noting that 2 is decomposed in  $K$  if and only if  $n$  is odd and  $p \equiv 1 \pmod{8}$ , we are left with the possibilities for  $\mathfrak{c}(\xi)$  as stated.  $\square$

**Lemma 5.2.**  $\mathfrak{c}(\xi_1) = \mathfrak{c}(\xi_2)$  is given in line  $\star$  of Proposition 5.1.

*Proof.* Let  $\xi$  be one of  $\xi_1$  or  $\xi_2$ . By Proposition 5.1 it suffices to show that the primes of  $K$  that divide 2 do not ramify in  $\mathfrak{c}(\xi)$ . So let  $\wp$  be a prime ideal of  $K$  that contains 2 and let  $R_\wp$  be the valuation ring in the completion  $K_\wp$  (so  $R_\wp \cong \mathbb{Z}_2$  if  $q \equiv 1 \pmod{8}$  and  $R_\wp \cong \mathbb{Z}_2[\zeta_3]$  if  $q \equiv 5 \pmod{8}$ ). By [13, Theorem VII.12 and Theorem VII.4] the image of  $R_\wp G$  in  $\mathrm{End}(K_\wp \otimes W)$  is isomorphic to

$$\Delta_\xi(R_\wp G) = \begin{pmatrix} R_\wp & 2R_\wp^{1 \times (q-1)/2} \\ R_\wp^{(q-1)/2 \times 1} & R_\wp^{(q-1)/2 \times (q-1)/2} \end{pmatrix}.$$

In particular the  $R_\wp G$ -lattices in  $K_\wp \otimes W$  form a chain

$$\dots \supset L' \supset L \supset 2L' \supset 2L \dots$$

with  $L'/L \cong R_\wp/2R_\wp$ . If  $F \in \mathcal{F}_G(W)$  is non-degenerate and  $L$  is  $G$ -invariant, then also its dual lattice is  $G$ -invariant. This shows that there is some  $F \in \mathcal{F}_G(W)$  such that  $L'$  is the dual lattice of  $L$ . But then  $Q_F(L) \subseteq R_\wp$  because otherwise the even sublattice of  $L$  would be a  $G$ -invariant sublattice of index 2 in  $L$ . So  $L$  is a semi-regular quadratic  $R_\wp$ -module in  $(K_\wp \otimes W, F)$  and by Corollary 3.5 this implies that  $\mathfrak{c}(K_\wp \otimes W, F) = 1$ .  $\square$

Note that for  $n = 1$  and  $n = 2$  it is also possible to deduce this lemma using the character theoretic method from [12] (see [2, Section 6.4]).

6. THE ORTHOGONAL CHARACTER TABLE OF  $\mathrm{SL}_2(2^n)$ 

We now assume that  $q = 2^n$  with  $n \geq 2$  and put  $G := \mathrm{SL}_2(q)$ . Then the ordinary character table of  $G$  is given in [5, Theorem 38.2]:

**Theorem 6.1** ([5, Theorem 38.2]). *Let  $\nu$  be a generator of  $\mathbb{F}_q^\times$  and consider the elements*

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad c := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad a := \begin{pmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{pmatrix}$$

*of  $G$ . The group also contains an element  $b$  of order  $q + 1$ . The character table of  $G$  is*

	$1_G$	$c$	$a^\ell$	$b^m$
$\mathbf{1}$	1	1	1	1
$\psi$	$q$	0	1	-1
$\chi_i$	$q + 1$	1	$\zeta_{q-1}^{i\ell} + \zeta_{q-1}^{-i\ell}$	0
$\theta_j$	$q - 1$	-1	0	$-\zeta_{q+1}^{jm} - \zeta_{q+1}^{-jm}$

where  $1 \leq i \leq \frac{q-2}{2}$ ,  $1 \leq j \leq \frac{q}{2}$ ,  $1 \leq \ell \leq \frac{q-2}{2}$ ,  $1 \leq m \leq \frac{q}{2}$ .

In contrast to the odd characteristic case all characters have totally real character field and Schur index 1.

**Theorem 6.2** (Orthogonal representations of  $\mathrm{SL}_2(2^n)$ ). *Let  $q = 2^n$ ,  $n \geq 2$  and  $G = \mathrm{SL}_2(q)$ . Then the non-trivial irreducible characters of  $G$  have  $G$ -invariant bilinear forms with the following algebraic invariants.*

Character	Invariant
$\psi$	$d_\pm(\psi) = q + 1$
$\chi_i, 1 \leq i \leq \frac{q-2}{2}$	$\mathfrak{c}(\chi_i) = \begin{cases} 1 \in \mathrm{Br}(\mathbb{Q}(\chi_i)) & \text{if } n \text{ is odd, see Theorem 6.4} \\ \text{see Theorem 6.3} & \text{if } n \text{ is even} \end{cases}$
$\theta_j, 1 \leq j \leq \frac{q}{2}$	$\mathfrak{c}(\theta_j) = \begin{cases} (-1, -1) \in \mathrm{Br}(\mathbb{Q}(\sqrt{5})) & \text{if } q = 4, \\ 1 \in \mathrm{Br}(\mathbb{Q}(\theta_j)) & \text{if } q \geq 8. \end{cases}$

*Proof.* For the Steinberg character  $\psi$  we again have that  $\psi + \mathbf{1}$  is the character of a 2-transitive permutation representation. In particular  $d_\pm(\psi) = d_\pm(\mathbb{A}_q) = q + 1$ . For the characters  $\theta_j$  of degree  $q - 1$  we note that the restriction of these characters to the normalizer  $B \cong C_2^n \rtimes C_{q-1}$  of the Sylow-2-subgroup of  $G$  is the character of an irreducible rational monomial representation  $V$ . So  $V$  has an orthonormal basis and hence  $\mathfrak{c}(\theta_j) = \mathfrak{c}(\mathbb{I}_q \otimes K)$  is given in Example 2.4.  $\square$

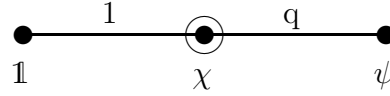
To describe the Clifford invariant of the characters  $\chi_i$  of degree  $q + 1$  note that for the infinite places of  $K$  the invariant  $[c(\chi_i) \otimes_K \mathbb{R}] \in \mathrm{Br}(\mathbb{R})$  is non-trivial if and only if  $q = 4$ , because in all other cases, the character degree is 1 (mod 8).

For the odd finite primes of  $K$ , the Clifford invariant of  $\chi_i$  is given in the next theorem:

**Theorem 6.3.** *Let  $1 \leq i \leq (q-2)/2$ ,  $K = \mathbb{Q}(\chi_i) = \mathbb{Q}[\vartheta_{q-1}^{(i)}]$ , and let  $\wp$  be some maximal ideal of  $\mathbb{Z}_K$  such that  $\wp \cap \mathbb{Z} = p\mathbb{Z}$  for some odd prime  $p$ . Then  $[c(\chi_i) \otimes K_\wp] \in \text{Br}(K_\wp)$  is not trivial if and only if*

*(i)  $p \equiv \pm 3 \pmod{8}$ , and (ii)  $(q-1)/(\gcd(q-1, i))$  is a power of  $p$ .*

*Proof.* We first note that condition (ii) implies that  $p$  divides  $q-1$ . If condition (ii) is not fulfilled, then the reduction of  $\chi_i$  modulo  $\wp$  is an irreducible Brauer character (see for instance [3]). In particular the orthogonal  $K_\wp G$ -module  $V$  affording the character  $\chi_i$  contains an (even) unimodular  $R_\wp$ -lattice. So Corollary 3.7 tells us that  $[c(\chi_i) \otimes K_\wp] = 1 \in \text{Br}(K_\wp)$ . If the condition (ii) is satisfied, then  $\wp$  is the unique prime ideal of  $K$  that contains  $p$ , the extension  $K_\wp/\mathbb{Q}_p$  is totally ramified, and (again by [3]) the  $\wp$ -modular Brauer tree of the block containing  $\chi_i$  is given as



where the multiplicity of the exceptional vertex  $\chi$  is  $\frac{p^a-1}{2}$  with  $a = \nu_p(q-1)$ . In particular [13, Theorem (VIII.3)] yields that the  $R_\wp$ -order  $R_\wp G$  acts on  $V$  as

$$\Delta_{\chi_i}(R_\wp G) = \begin{pmatrix} R_\wp & \wp R_\wp^{1 \times q} \\ R_\wp^{q \times 1} & R_\wp^{q \times q} \end{pmatrix}.$$

As in the proof of Lemma 5.2 the  $R_\wp G$ -invariant lattices in  $V$  form a chain:

$$\dots \supset L' \supset L \supset \wp L' \supset \wp L \dots$$

with  $L'/L \cong R_\wp/\wp R_\wp$ . So there is a  $G$ -invariant form  $F$  on  $V$  such that  $L' = L^\#$ , in particular the  $\wp$ -adic valuation of the determinant of  $L$  is 1. Choose  $(b_1, \dots, b_q) \in L^q$  such that the images form a basis  $\overline{B}$  of  $L/\wp L'$  and put  $W := \langle b_1, \dots, b_q \rangle_{K_\wp} \leq V$ . The modular representation  $L/\wp L'$  is isomorphic to the  $\wp$ -modular reduction of the Steinberg module  $\psi$ . In particular the determinant of the Gram matrix of  $\overline{B}$  is  $\overline{q+1} \in \mathbb{Z}/p\mathbb{Z} \cong R_\wp/\wp R_\wp$ . As  $\wp$  is odd and  $q+1 \in R_\wp^\times$  this gives the discriminant of the bilinear  $K_\wp$ -module

$$d_\pm(W, F|_W) = (q+1)(K_\wp^\times)^2 = 2(K_\wp^\times)^2$$

because  $q+1 \equiv 2 \pmod{p}$  since  $p$  divides  $q-1$ . We can now apply Corollary 3.8 to conclude that the Clifford invariant of  $(V, F)$  is non-trivial, if and only if 2 is not a square in  $K_\wp$ , if and only if 2 is not a square in  $\mathbb{F}_p = R_\wp/\wp$  which is equivalent to condition (i) by quadratic reciprocity.  $\square$

**Theorem 6.4.** *If  $q = 2^n$  and  $n$  is odd then  $\mathfrak{c}(\chi_i) = 1 \in \text{Br}(\mathbb{Q}(\chi_i))$  for all  $1 \leq i \leq \frac{q-2}{2}$ .*

*Proof.* Let  $M := \mathbb{Q}_2[\zeta_{2^n-1}]$  be the unramified extension of  $\mathbb{Q}_2$  of degree  $n$ . Then  $M$  is a splitting field for  $G$ . Moreover the  $M$ -representation  $V_M$  affording the character  $\chi_i$  is induced up from a linear  $M$ -representation of the normalizer  $B = C_2^n \rtimes C_{2^n-1}$  of the Sylow-2-subgroup of  $G$ . In particular  $V_M$  is an irreducible monomial representation and hence the standard form  $F_M$  is  $G$ -invariant,

so  $(V_M, F_M) \cong \mathbb{I}_{2^n+1} \otimes M$ . For  $n \geq 3$  the dimension of  $V_M$  is  $\equiv 1 \pmod{8}$  and so by Example 2.4 the Clifford invariant of  $(V_M, F_M)$  is trivial in  $\text{Br}(M)$ . Now let  $K = \mathbb{Q}(\chi_i)$ ,  $(V, F)$  an orthogonal  $KG$ -module affording the character  $\chi_i$ , and let  $\wp$  be some prime ideal of  $K$  dividing 2. As  $K \subseteq \mathbb{Q}[\zeta_{2^n-1}]$  the completion of  $K$  at  $\wp$  is contained in  $M$  and, by the same argument as before,  $(V \otimes M, F) \cong (V_M, aF_M)$  for some non-zero  $a \in M$ . In particular  $\mathfrak{c}(V \otimes M, F) = 1$  in  $\text{Br}(M)$ . As  $[M : \mathbb{Q}_2] = n$  is assumed to be odd, also  $[M : K_\wp]$  is odd and hence  $\mathfrak{c}(V \otimes K_\wp, F) = 1$  in  $\text{Br}(K_\wp)$ . This argument shows that no even prime  $\wp$  of  $K$  ramifies in  $c(V, F)$ . Also the real primes do not ramify because  $\dim(V) \equiv 1 \pmod{8}$ . So by Theorem 6.3 there is at most one prime ideal of  $K$  that ramifies in  $c(V, F)$ . But the number of ramified primes is even, which shows that  $\mathfrak{c}(\chi_i) = 1$  in the Brauer group of  $K$ .  $\square$

Note that Theorem 6.4 together with Theorem 6.3 implies the well known fact that if  $n$  is odd then all primes  $p$  dividing  $2^n - 1$  satisfy  $p \equiv \pm 1 \pmod{8}$  (because then  $2^{(n+1)/2}$  is a square root of 2 modulo  $p$ ).

**Remark 6.5.** *In the situation of Theorem 6.3 if  $[c(\chi_i) \otimes K_\wp] \in \text{Br}(K_\wp)$  is non-trivial and  $q \neq 4$ , then an odd number of even primes of  $K$  also ramify in  $c(\chi_i)$ . However, we did not determine in general which even primes of  $K$  ramify in  $c(\chi_i)$  for the case that  $n$  is even. Of course the same argument as in the proof of Theorem 6.4 works if the primes above 2 are decomposed in  $\mathbb{Q}(\zeta_{q-1}^i)/\mathbb{Q}(\wp_{q-1}^{(i)})$ .*

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